BRST
&
Deformation Quantization

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Abstract

The four fundamental interactions in nature are strong, electromagnetic, weak and gravity. All of them are gauge field theories with different gauge groups. Quantizing gauge theories requires extra care as there are unphysical degrees of freedom that need to be modded out. The cleanest way by far to perform quantization of gauge theories is via a homological method called BRST. In this report we will explore the origins of gauge symmetries, their geometric and algebraic structure and demonstrate the power of BRST to facilitate quantization. We will also discuss another formal quantization scheme called deformation quantization which deforms the Poisson algebra of functions on a symplectic manifold to a non-commutative star algebra.

Contents

1 Introduction: the problem of quantization of non-abelian gauge theories

The genesis of BRST was in the problem of trying to quantize non-abelian gauge theories. Gauge theories are constrained systems, and so if one naively tries to quantize them using the canonical Hamiltonian/Lagrangian, one runs into inconsistencies like non-zero probability amplitude for states with unphysical polarization, leading to violation of unitarity of the S-matrix. The problem can be equivalently phrased in terms of need for appropriate (gauge invariant) path integral measure. It posed great challenge in the quantization of non-abelian gauge theories viz. Yang Mills theories in the 1960s. The Standard Model itself describes gauge theories with both abelian (U(1)) and non-abelian (SU(2), SU(3)) gauge symmetries. This makes path-integral quantization of non-abelian gauge systems extremely relevant for understanding the fundamental physics underlying the universe we live in. Faddev and Popov [FP67] came up with a prescription to fix this problem once and for all in 1967. In the beginning what looked like a mere trick of introducing anticommuting "ghosts" to fix the path-integral measure, turned out to be encoding a completely new global fermionic symmetry called BRST.

Let us go through the steps that Faddeev and Popov took to derive the correct gauge-fixed action. We will be schematic with the equations to highlight the key structural aspects of the Faddeev-Popov procedure. For details, one can refer to really good resources like [PS95, Wei96].

- We start with the path integral for a pure gauge field

$$Z_{full}[A] = \int [DA] e^{iS_0[A]} = \int [D\widetilde{A}][Dg] e^{iS_0[A]} \tag{1}$$

  where we have split the measure along base and vertical of the principal G-bundle of field configurations.

- The path-integration above includes field configurations along gauge orbits as well. This has to be modded out. To that end, we first compute the volume of the gauge orbit corresponding to gauge transformations that preserve some gauge condition $\mathcal{F}(A) = 0$. Let A^* be a solution of the guage condition $\mathcal{F}(A)$. We have

$$\begin{aligned}\mathrm{Vol}_{\mathcal{F}}(A^*) &= \int [Dg]\ \delta(\mathcal{F}({}^gA^*)) \\ &= \int \frac{[D\mathcal{F}({}^gA)]}{\det\left(\frac{\delta\mathcal{F}({}^gA^*)}{\delta g}\right)} \delta(\mathcal{F}({}^gA^*)) \\ &= \frac{1}{\det\left(\frac{\delta\mathcal{F}({}^gA^*)}{\delta g}\right)} =: \Delta_{\mathcal{F}}^{-1}(A^*)\end{aligned} \tag{2}$$

  where $\delta(\mathcal{F}({}^gA^*))$ selects all gauge transformations that preserves the gauge fixing condition and ${}^gA = g^{-1}Ag + gdg^{-1}$ is the action of g on A.

  $\Delta_{\mathcal{F}}(A)$ is called the Faddeev-Popov determinant and it captures the inverse volume of the gauge orbit preserving a gauge condition $\mathcal{F}$ in the field configuration space. Note that $\Delta_{\mathcal{F}}(A)$ is invariant under action of g

$$\Delta_{\mathcal{F}}(A) = \Delta_{\mathcal{F}}({}^gA)$$

  since the integration measure is invariant under action of g.

- We are now ready to define the Faddeev-Popov gauge-fixed path-integral. It is just the full path integral in (1) divided by the volume of gauge orbit

$$
\begin{aligned}
Z_{gf}[A] = \frac{Z_{full}[A]}{\int[Dg]} &= \frac{1}{\int[Dg]}\int[DA][Dg']\ \Delta_{\mathcal{F}}(A)\ \delta(\mathcal{F}({}^{g'}\!A))e^{iS_0[A]} \qquad (3)\\
&= \frac{1}{\int[Dg]}\int[DA][Dg']\ \Delta_{\mathcal{F}}({}^{g'}\!A)\ \delta(\mathcal{F}({}^{g'}\!A))e^{iS_0[{}^{g'}\!A]} \qquad (4)\\
&= \frac{\cancel{\int[Dg']}}{\cancel{\int[Dg]}}\int[DA]\ \Delta_{\mathcal{F}}(A)\ \delta(\mathcal{F}(A))e^{iS_0[A]} \qquad (5)\\
&= \int[DA]\ \underbrace{\Delta_{\mathcal{F}}(A)}_{\text{Faddeev-Popov term}}\ \underbrace{\delta(\mathcal{F}(A))}_{\text{gauge-fixing term}}\ e^{iS_0[A]} \qquad (6)\\
&= \int[DA][Dc][D\bar{c}][Db]e^{i(S_0[A]+S_{FP}[c,\bar{c}]+S_{gf}[b])} \qquad (7)
\end{aligned}
$$

where

$$
S_{FP}[c,\bar{c}] = \int d^4x\ i\bar{c}\left(\frac{\delta\mathcal{F}({}^{g}\!A)}{\delta g}\right)c = \int d^4x\ i\bar{c}\ \delta_c\mathcal{F}(A), \quad S_{gf}[b] = \int d^4x\ b\mathcal{F}(A) \qquad (8)
$$

and $\frac{\delta\mathcal{F}({}^{g}\!A)}{\delta g}$ is the Faddeev-Popov operator and $\delta_c\mathcal{F}(A)$ stands for the variation of the gauge condition $\mathcal{F}(A) = 0$ under a gauge transformation with parameter the ghost field c.

Some explanations in the derivation above are in order. In (3), we have expanded unity inside the path integral using (2). In (4), we have used gauge invariance of $\Delta_{\mathcal{F}}(A)$ and $S_0[A]$ to replace A with ${}^{g'}\!A$. In (5), we have used gauge invariance of the measure $[DA]$ to shift all ${}^{g'}\!A$ back to A. Finally, in (7) we re-express the Faddeev-Popov determinant $\Delta_{\mathcal{F}}(A)$ and the gauge condition $\delta(\mathcal{F}(\mathcal{A}))$ as path integrals over anticommuting fields $c, \bar{c}$ and a bosonic field b with actions S_{FP} and S_{gf} respectively.

Thus we see that the gauge-fixed path integral has three additional fields $c, \bar{c}, b$ with the ghost fields $(c, \bar{c})$ helping to mod out the gauge orbits and the auxiliary field b selecting a gauge slice i.e. a section of the principal G-bundle of field configurations.

Let us now turn to the gauge invariance of the resulting gauge-fixed action. Even though we have gauge-fixed, the action still retains all the gauge symmetries albeit packaged into a single global fermionic symmetry aka BRST. Let us look at the total action

$$
S_{tot}[A, c, \bar{c}, b] = S_0[A] + S_{FP}[c, \bar{c}, A] + S_{gf}[b, A]
$$

While S_0 is gauge-invariant (hence BRST invariant), S_{FP} and S_{gf} individually are not. However, the sum $S_{FP}[c, \bar{c}, A] + S_{gf}[b, A]$ turns out to be BRST exact.

$$
S_{FP} + S_{gf} = \delta_\Omega\Psi
$$

where Ψ is called the gauge-fixing fermion and Ω is the BRST charge. If we are able to establish nilpotency $\delta_\Omega^2 = 0$, it would automatically imply BRST invariance of the total action S_{tot}.

The task now is to prove that the sum $S_{FP} + S_{gf}$ is indeed BRST exact corresponding to some nilpotent BRST transformation. Firstly, the nilpotent BRST transformation corresponding to a gauge group whose Lie algebra structure constants are f^a_{bc} looks like

$$
\begin{gathered}
\delta_\Omega A^a_\mu = (D_\mu c)^a = \partial_\mu c^a - gf^a_{bc}A^bc^c, \quad \delta_\Omega c^a = \frac{g}{2}f^a_{bc}c^bc^c, \\
\delta_\Omega\bar{c}_a = ib_a, \quad \delta_\Omega b_a = 0
\end{gathered} \qquad (9)
$$

It is easy to check nilpotency. While $\delta_\Omega^2 \bar{c}_a$ is trivially zero, $\delta_\Omega^2 c^a = 0$ follows from anticommuting property of the ghost field c^a's and the Jacobi identity of structure constants. In a similar fashion, one can show that $\delta_\Omega^2 A_\mu^a = 0$ - the terms proportional to $fc\partial c$ cancel while the terms proportional to $ffAcc$ die by virtue of anticommutativity of c's and Jacobi identity of structure constants. If we now choose gauge-fixing fermion Ψ to be

$$\Psi = \int d^4x \; i\bar{c}_a \mathcal{F}^a(A) \tag{10}$$

then we produce exactly the Faddeev-Popov ghost term and the gauge-fixing term

$$\delta_\Omega \Psi = \int d^4x (-i\bar{c}_a \delta_c \mathcal{F}^a - b_a \mathcal{F}^a) \tag{11}$$

which is in complete agreement with (8). Thus we have shown the existence of BRST symmetry in the gauge-fixed Faddeev-Popov action for a non-abelian gauge theory.

Now that we understand the role of BRST in quantization of non-abelian gauge theories, it is interesting to think about how general the symmetry itself is and if it has any classical interpretation. That's exactly what we are about to do in the next section by first understanding constraints and how they beget gauge symmetries.

2 Constraints and BRST

Constraints are intimately connected to gauge symmetries that are in turn described by BRST. In the following sections we provide a concise review of the key ideas starting from constraints in classical systems and leading upto BRST. The review is based on [dir64, HT92, FO06, Kim93, FL94].

2.1 Origin of gauge symmetries: constrained momenta

The two most prominent ways to describe dynamics of classical systems are the Lagrangian and Hamiltonian formalisms. The equations of motions in the two formalisms are equivalent for systems without constraints (on momentum). However, if the momenta are constrained (or equivalently a *singular* Lagrangian's Hessian admits zero modes) then the map from Lagrangian $L(q, \dot{q}, t)$ to Hamiltonian $H(q, p, t)$ becomes many to one. The standard form of Euler-Lagrange (E-L) equations is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j} = \frac{\partial L}{\partial q^j}.$$

For constraint analysis, it will be more useful to pass to the Hessian of L with respect to the velocities $\dot{q}^i$. We use chain rule and express the time derivative on the left in terms of $q^i, \dot{q}^i$ derivatives

$$\text{Hess}(L)_{ij}\ddot{q}^i = V_j,$$

where

$$\text{Hess}(L)_{ij}(q, \dot{q}) := \frac{\partial^2 L}{\partial \dot{q}^i \dot{q}^j} = \frac{\partial p_j}{\partial \dot{q}^i}$$
$$V_j(q, \dot{q}) := \frac{\partial L}{\partial q^j} - \dot{q}^i \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j}.$$

We can see that the existence of a unique acceleration rests on the invertibility of the Hessian. If the Hessian has zero modes, there will be relations among the momenta or equivalently the velocities will not be a unique function of position and momenta. In a nutshell

$$\text{null space of Hess}(L) \equiv \text{constraints on momenta}$$

We can make the relationship more precise. Let $n^i(q,\dot{q})$ be a zero mode i.e. it is a null direction for Hess(L). Then we have a pair of constraints

$$\begin{aligned} n^j(q,\dot{q})\frac{\partial p_j}{\partial \dot{q}^i}(q,\dot{q}) = 0 \implies n^j(q,\dot{q})dp_j|_{q=const} &= 0 \quad \text{off-shell constraint} \\ n^j(q,\dot{q})V_j(q,\dot{q}) &= 0 \quad \text{on-shell constraint} \end{aligned}$$

This is the starting point of the *Dirac-Bergmann* constraint analysis [dir64, Dir50, AB51, BG55]. An integrated version of the above off-shell constraint per null mode of Hess(L) is called a *primary constraint*. They are functions of q, p only and have no explicit $\dot{q}$ dependence.

$$\int n^j dp_j|_{q=const} = 0 \implies \phi(p,q) = 0$$

These primary constraints must be preserved in time, which lead to on-shell *secondary* constraints.

$$\frac{d^n\phi}{dt^n} = 0 \implies \phi^{(n)} \equiv \{\{\phi, H\}, H\}, \ldots, H\} = 0 \tag{12}$$

The above process of differentiating old constraints to get new constraints stops after a finite number of steps (since we have finite number of degrees of freedom to begin with). The constraints so obtained generate a *constraint ideal* defined as the subalgebra of functions in phase space that vanish weakly (onshell)

$$I = \{f \in C^\infty(M) | f \approx 0\}$$

This ideal structure is mathematically very important as it allows us to define a quotient of algebra of functions. Quotienting will be crucial to understanding BRST in the later sections.

Another key feature of many singular systems is the presence of arbitrary time dependent functions in the solutions to the equation of motion. To see this, first note that the singular character of the system allows one to modify the canonical Hamiltonian ($H := p_i\dot{q}^i - L$) to an extended Hamiltonian without affecting the equations of motion

$$H_{ext} := H + u^i(p,q)\phi_i(p,q) \tag{13}$$

where ϕ_i are constraints and $u^i(p,q)$ are corresponding Lagrange multipliers. These Lagrange multipliers can be understood to be enforcing the constraints inside the action

$$S_{ext}[p(t), q(t), u(t)] = \int dt\, (p\dot{q} - H_{ext})$$

and at the same time making the Legendre transformation $(q,\dot{q}) \to (p,q,u)$ invertible. The Lagrange multipliers u turn out to be constrained as well. They satisfy the following system of consistency equations

$$\{\phi_i, H_{ext}\} \approx 0 \implies \{\phi_i, H\} + u^j\{\phi_i, \phi_j\} \approx 0 \tag{14}$$

The most general solution to the above linear system will be a sum of a particular and the general solution of the associated homogeneous system

$$u^j = u^j_{part} + u^j_{hom} = u^j_{part} + c^a v^j_a \tag{15}$$

where v_a is a solution to the homogeneous system

$$v^j_a\{\phi_i, \phi_j\} \approx 0 \tag{16}$$

Note the index i labels a generator of the constraint ideal while a labels a solution to the homogeneous system. We are now ready to write down the equations of motion of the constrained system

$$\dot{f} \approx \{f, H_{ext}\} \approx \{f, H'\} + c^a \{f, \phi_a\} \tag{17}$$

$$\phi_i \approx 0 \tag{18}$$

where $H' = H + u^j_{part}\phi_j$ and $\phi_a = v^j_a \phi_j$. The c^a in the above equation are the completely arbitrary time-dependent functions that characterize the *gauge symmetries* of the system. While the physics of such systems is unaffected by a particular choice of these functions, one can break some or all of the gauge symmetries by fixing these functions (gauge fixing). In any case, gauge fixing can be thought of as a choice of "reference frame". We don't completely gauge fix unless absolutely necessary - spoils manifest symmetries, possible obstruction to global gauge etc.

Now that we have seen how gauge symmetries arise, we would like to characterize the constraints in the constraint ideal that are responsible for them. It will turn out that not all constraints can generate gauge transformations, only a subalgebra generated by so called *first class constraints*.

2.2 Generator of gauge transformations: first class constraints

In the previous section we found that constraints hit twice, once as constraints on phase-space coordinates and next as constraints on Lagrange multipliers.

1. **Constraints on phase-space coordinates** (p, q): null space of $\text{Hess}(L)_{ij}$ + time derivatives aka primary and secondary constraints. This is really then just the constraint ideal I.
2. **Constraints on Lagrange multipliers** u: null space of matrix of Possion bracket of constraints $\{\phi_i, \phi_j\}$ or equivalently the null space of $\{\phi_i, .\}$ inside I. These new constraints capture relations among constraints in the constraint ideal I.

We were also able to identify the latter as the one responsible for gauge symmetries. This observation motivated Dirac to classify all phase-space functions based on their relationship with constraints when taking Poisson brackets, leading to the notion of *first class functions*. First class functions are phase space functions that commute weakly with all constraints. This means one can intuitively think of them as "constraints among constraints". Mathematically speaking, they are just functions in the *Poisson normalizer* $N(I)$ of the constraint ideal I [Kim93]

$$\{N(I), I\} \subseteq I.$$

We now make a few observations that help us better characterize the relationship between constraints and first class functions.

- First class functions in $N(I)$ generate a Poisson subalgebra of $C^\infty(M)$. This means it closes under Lie bracket. This is a simple consequence of Jacobi identity

$$\{\{N(I), N(I)\}, I\} = -\{\{N(I), I\}, N(I)\} - \{\{I, N(I)\}, N(I)\} \approx 0$$

- First class constraints generate the Poisson ideal $I' = N(I) \cap I$ of $N(I)$.

$$\{I', N(I)\} \subseteq I'$$

  This follows from the previous property and the definition of $N(I)$. These are relations among constraints that also happen to belong to the constraint ideal I. We will often refer to I' as the first class constraint algebra/ideal.

- We further note that H_{ext}, H' and ϕ_a in (17) are all first class. The first two follow from (14) while the third is just a restatement of (16).

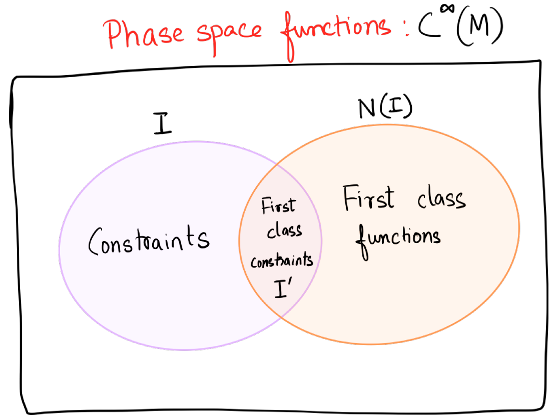

Figure 1: Relationship between constraint ideal I, Poisson normalizer $N(I)$ and first class constraint ideal I'.

We thus conclude that $\phi_a \in I'$ are nothing but first class constraints that generate gauge transformations in the second term of (17)

$$\delta_{gauge} = c^a\{\phi_a, .\}$$

The arbitrary functions c^a coming from the Lagrange multipliers correspond to gauge parameters associated to the gauge transformation generated by these first class constraints.

We now turn to *second class* constraints, which are simply constraints that are not first class. Thus a second class constraint f will not weakly commute with atleast one generating constraint in the constraint ideal

$$\exists\, i \quad \text{such that } \{f, \phi_i\} \not\approx 0$$

Failure to commute weakly means they don't generate gauge transformations and are rather indicative of *inessential* degrees of freedom. They come in conjugate pairs, and setting them to be weakly equal to zero is equivalent to setting them to be strongly equal to zero. This was Dirac's idea of killing the second class constraint degrees of freedom and leads to the notion of Dirac bracket. The Dirac bracket modifies the Poisson bracket so that it doesn't generate flows due to second class constraints (that can take one outside the constraint surface). One then quantizes the Dirac bracket instead of the Poisson bracket.

We saw that constraints on Lagrange multipliers are encoded by the matrix of Poisson brackets of a generating set of constraints C_{ij}.

$$C_{ij} = \{\phi_i, \phi_j\} = \begin{pmatrix} 0 & 0 \\ 0 & C_{\alpha\beta} \end{pmatrix}$$

such that $C_{\alpha\beta}$ is antisymmetric and $\det C_{\alpha\beta} \neq 0$. The split structure in C_{ij} seprates the two classes of constraints: zero block corresponding to first class constraints and the invertible (non-singular) block corresponding to the second class constraints. Since an anti-symmetric matrix is invertible iff it is of even size, we have that second class constraints are always even in number. The Dirac bracket itself crucially depends on this invertibility

$$\{f, g\}_{DB} = \{f, g\} - \{f, \phi_\alpha\}C^{\alpha\beta}\{\phi_\beta, g\}$$

The purpose of the second term above is to kill the effect of Poisson bracket in the ϕ_α directions.

While Dirac bracket tells us how to deal with second class constraints, quantization of Dirac bracket itself is challenging. This is because one is faced with the task of having to find operator representation of the non-linear terms in the second factor above. One way around this is to not use Dirac bracket at all and try to lift second class to first class. The idea is to view second class constraints as first class in a larger gauge theory. This approach is referred to as the *BFF embedding/conversion* [BFF89]. The second class constraints are expanded in the extra gauge degrees of freedom whose number match the number of second class constraints. Then the second class constraints can be abelianized in the extended phase space and hence converted into first class.

If one views ordinary (unconstrained) dynamics as trajectories on a manifold, it is interesting to ask what "trajectories" look like for constrained dynamics on a manifold. The next section is devoted to characterizing the geometry of constrained dynamics. As a bonus, we will eventually be led to an extremely intuitive way to think about classical BRST.

2.3 Geometry of constraint surface: null foliation of characteristic distribution

For constrained systems, phase space dynamics is restricted to only a portion of the full phase space M where constraints I are satisfied. We call the zero set of all constraints (first class + second class) the *constraint surface/submanifold* (provided zero is a regular value).

$$M_0 = Z(I)$$

As we discussed before, first class constraints beget gauge symmetries. How do they manifest geometrically? The answer is *foliation by null surfaces* [HT92].

Firstly, Hamiltonian vector fields corresponding to first class functions are always along the constraint surface (and hence restrict to M_0). This is because as derivations they preserve the constraint ideal I.

$$X_f(I) = \{f, I\} \subseteq I, \quad f \in N(I)$$

where X_f is the Hamiltonian vector field corresponding to the phase-space function f. Furthermore, first class constraints generate a Poisson ideal I' of $N(I)$ and hence the corresponding Hamiltonian vector fields also restrict to M_0 and form a Lie algebra. In fact we have a Lie algebra homomorphism between the first class constraint ideal I' and the corresponding Hamiltonian vector fields

$$[X_f, X_g] = X_{\{f,g\}}, \quad f, g \in I'$$

This closure property implies involutivity of the distribution spanned by the Hamiltonian vector fields corresponding to the first class constraints. By Frobenius integrablity theorem, we have foliation by maximal integral submanifolds. The dimension of a leaf is same as the number of independent first class constraints. Secondly, the "null" property of these leaves is the fact that the symplectic 2-form is completely degenerate on the tangent space of these leaves.

$$\omega(X_f, X_g) = \{f, g\} \approx 0, \quad f, g \in I'$$

In short we have an involutive distribution $D(I')$ associated to the first class constraint algebra that is null over M_0 called the *characteristic distribution.*

$$D(I') = TM_0 \cap TM_0^{\perp}$$

where by $\perp$ we mean symplectic complement. TM_0 is spanned by Hamiltonian vector fields due to $N(I)$, while Hamiltonian vector fields due to I lie in the symplectic complement $TM_0^{\perp}$. We should point two extreme cases that often occur in practice viz. if all constraints are first class or if all are second class.

- In the special case where $I \subseteq N(I) \iff I' = I$ i.e. all constraints are first class, we say that a first class constraint surface M_0 is a *coisotropic submanifold* of the ambient symplectic manifold. Coisotropic refers to the fact that $\{I, I\} \subseteq I \iff TM_0^\perp \subseteq TM_0$. The process of reduction of phase space of such systems is called *coisotropic reduction*.

- When all constraints are second class, the pullback of the symplectic 2-form to the constraint manifold $\omega|_{M_0} = i^*\omega$ is non-degenerate and the null distribution $D(I') = 0$. Hamiltonian flow due to constraints are transverse to TM_0. The constraint surface M_0 is then a *symplectic submanifold*. The first class constraint algebra is $I' = I^2$ since $di|_{M_0} = 0$ for all $i \in I'$. The corresponding reduction of phase space is called *symplectic reduction*.

We are now ready to describe the two step process of *reduction* that takes us to the reduced phase space. Geometrically speaking, this allows one to pass from the original Poisson manifold M to the constraint submanifold M_0 and finally to the leaf space orbifold $\widetilde{M}$.

1. Restriction to the constraint surface : both first class & second class constraints are used

$$M \to M_0 = Z(M)$$

2. Quotienting out leaves of null foliation : only first class constraints are used

$$M_0 \to \widetilde{M}, \quad T\widetilde{M} \cong \frac{TM_0}{D(I')}$$

Putting them together

$$M//G := Z(M)/G \quad \textit{Poisson manifold reduction}$$

where $G = \exp(\mathrm{Ham}(I'))$ is the group of Hamiltonian flows generated by the first class constraints aka the gauge group.

The above geometric procedure implementing *restriction+quotient* is at the heart of reduction of constrained systems including gauge systems. This geometric picture also strongly suggests a dual homological interpretation aka classical BRST that will be called discussed next.

2.4 Classical BRST: algebraic analogue of phase-space reduction

It turns out the BRST symmetry that was discovered in the quantum world, has a classical incarnation as explained beautifully in [FO90, FOK91, Kim93, FO06]. We will sketch how Poisson manifold reduction can be equivalently described as a reduction of the corresponding Poisson algebra. We will also comment on how this point of view affords an extremely useful generalization.

The algebraic (Gelfand) dual of a manifold M is the space of functions on it encoded by the *structure sheaf* $\mathcal{O}_M$. Instead of focussing on how the manifold gets reduced, we turn to the reduction of algebra of functions on it aka classical observables. Given the algebra of observables $C^\infty(M)$ on M and a vanishing constraint ideal I, one wants to perform the following two back-to-back quotients on the algebra of functions:

1. Restriction to constraint surface M_0

$$C^\infty(M) \to C^\infty(M_0) \cong \frac{C^\infty(M)}{I}$$

The above isomorphism is because functions on M_0 can be identified as long as they differ by functions vanishing on M_0.

2. Restriction to leaf space $\widetilde{M}$

$$C^\infty(M_0) \to C^\infty(\widetilde{M}) \cong \frac{N(I', I)}{I} \cong \frac{N(I)}{I'}$$

where $N(I', I) = \{f \in C^\infty(M) | \{f, I'\} \subseteq I\}$ is the normalizer relative to the first class constraint ideal I' and the constraint ideal I [Kim93].

The first isomorphism above is because of the fact that functions on $\widetilde{M}$ correspond to functions on M that are constant along leaves or equivalently $\{f, I'\} \subseteq I$ ($\{I', .\}$ correspond to derivations tangent to the leaves). To prove the second isomorphism requires some work [Kim93].

This algebraic formulation is quite powerful as it continues to make sense even if the geometry is singular. Instead of talking about reduction of Poisson manifold M, one talks about the reduction of Poisson algebra $\mathcal{P}$ by some constraint Poisson ideal J [Kim93].

$$\mathcal{P}//J := \frac{N(J)}{J'} \quad \textit{Poisson algebra reduction}$$

where $J' = N(J) \cap J$ and $N(J)$ is the Poisson normalizer of J in $\mathcal{P}$.

Now that we are in the algebraic realm, we can package the double quotient process as computation of (co)homology. The process of reduction now get's substituted by the construction of a super-Poisson bicomplex whose zeroth degree cohomology contains all functions on the reduced phase-space. The two parts of the bicomplex are

- *Koszul-Tate complex* - implements restriction via (projective) resolution of $C^\infty(M_0)$ to the complex $\Lambda\mathfrak{g} \otimes C^\infty(M)$ of ghost momenta (sometimes also called anti-ghosts) and phase space functions. $\mathfrak{g}$ stands for the Lie algebra of the Hamiltonian vector fields due to first class constraints. It is negatively graded.
- *Chevalley-Eilenberg complex* - implements quotient down to the leaf space algebra $C^\infty(\widetilde{M})$ via Lie algebra cohomology of the bicomplex $(\Lambda\mathfrak{g} \otimes C^\infty(M)) \otimes \Lambda\mathfrak{g}^*$ of Koszul-Tate complex and ghost algebra. It is positively graded.

In the next few sections we explain the role of the bicomplex in a bit more detail.

2.4.1 Koszul-Tate (KT) dg algebra: $K = (\Lambda^\cdot\mathfrak{g} \otimes C^\infty(M), \delta)$

The purpose of this differential-graded (dg) algebra is to ultimately help reduce phase-space functions by constraints in constraints ideal I. But what is crucial is the choice of generating set of constraints. We want the generating set to be be a *regular sequence* i.e. the generators must be independent of each other. This clean separation of constraints into independent ones is technically referred to as *resolution* aka unquotienting.

We already know that first class constraints form a Poisson subalgebra that can thought of as a Lie algebra of Hamiltonian vector fields (or Poisson derivations) $\mathfrak{g}$, in which case picking a generating set is quite easy. However, if there are second class constraints in addition to first class the constraint algebra is not a Lie algebra and we do not have a canonical choice of free generating set. We have to then resolve $C^\infty(M_0)$ to some free $C^\infty(M)$ module step by step. The coarsest resolution of $C^\infty(M_0)$ is of course the quotient of $C^\infty(M)$ by the vanishing ideal of constraints I. However, I itself might not be freely generated i.e. there could be relations among its generators. So we have to resolve these relations until there are no more relations between relations. This process yields an "almost" acyclic sequence.

$$\ldots \xrightarrow{\delta} \Lambda^2\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} C^\infty(M) \longrightarrow 0$$

The action of the Koszul differential δ is quite straightforward. It annihilates phase space coordinates and pairs constraint generators with constraint functions.

- On phase-space coordinates z^i: $\delta(z^i) = 0$. This ensures $(\mathrm{Ker}\,\delta)_0 = C^\infty(M)$, where the subscript 0 indicates 0 ghost number.
- On constraint generators aka ghost momenta b_i: $\delta(b_i) = -\phi_i$, where ϕ_i is the corresponding first class constraint. This ensures $(\mathrm{Im}\,\delta)_0 = I$.

The homology of the Koszul complex is then by construction concentrated only at degree 0 [HT92].

$$H_p(K^\cdot) \cong \begin{cases} \frac{C^\infty(M)}{I} \cong C^\infty(M_0), & p = 0 \\ 0, & otherwise \end{cases}$$

2.4.2 Chevalley-Eilenberg (CE) dg algebra: $C = (K \otimes \Lambda^\cdot \mathfrak{g}^*, d)$

This dg algebra is responsible for modding out gauge orbits. The gauge symmetries are generated by Hamiltonian vector fields corresponding to the first class constraints. The dual to these vector fields are "vertical" 1-forms that are non-zero only along gauge orbits. These act as "ghosts" in the extended phase space as they represent non-physical degrees of freedom with opposite statistics. Although the exterior algebra of these 1-forms together with the exterior derivative is the simplest possible Chevalley-Eilenberg dg algebra, we can pass to a larger complex by tensoring it with a Koszul-Tate module K. This means that the complex now contains vector fields $\in \mathfrak{g}$ and their dual 1-forms $\in \mathfrak{g}^*$ as well phase-space functions $\in C^\infty(M)$.

What about the action of the differential operator for CE dg algebra? Let X_i and α^i be canonical dual bases for $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Using the two natural operations on $\Lambda\mathfrak{g}^*$ viz.

- contraction with vector: $\iota(X)\alpha = \alpha(X)$
- wedge with 1-form: $\varepsilon(\alpha)\omega = \alpha \wedge \omega$

we can construct the CE differential operator [FO06]

$$d = \varepsilon(\alpha^i)\rho(X_i) - \frac{1}{2}\varepsilon(\alpha^i)\varepsilon(\alpha^j)\iota([X_i, X_j]),$$

where ρ corresponds to the action of Lie algebra on the Koszul-Tate module. We define the ghost and ghost momenta to be the image of the canonical basis under the two operations: $c^i = \varepsilon(\alpha^i)$ are the ghosts and $b_i = \iota(X_i)$ are the ghost momenta. This leads to the familiar looking formula for the CE differential in the simplest case where constraints close according to a group

$$d = c^i\rho(X_i) - \frac{1}{2} f_{ij}^k c^i c^j b_k, \tag{19}$$

where f_{ij}^k are structure constants in $[X_i, X_j] = f_{ij}^k X_k$. It is no surprise we first encountered an operator of this kind when discussing Yang-Mills gauge theory in the introduction. The CE differential operator is getting us pretty close to the final BRST operator, all we need to do now is to combine d and δ.

2.4.3 BRST complex: $\mathcal{C} = (C, D)$

For the BRST bicomplex, we expose the positive and negative grading of the ghost and ghost momenta pieces in the CE dg algebra, $C^{p,q} = \Lambda^q\mathfrak{g} \otimes C^\infty(M) \otimes \Lambda^{*p}\mathfrak{g}$. The total degree (ghost number) of the complex is then given by sum of the positive and negative gradings, $n = p - q$. Thus degree n term of the complex is

$$\mathcal{C}^n = \oplus_{p-q=n} C^{p,q}.$$

The two differentials anticommute and we now compute the cohomology of the bicomplex wrt the total differential

$$D = d + (-1)^p \delta.$$

Note that the sign factor in front of δ is to ensure nilpotency of D. Finally, the cohomology wrt to the total differential is equivalent to CE Lie algebra cohomology with coefficients in the Koszul-Tate module. More precisely, one can show [FO90] that in degree n the cohomology of the total complex wrt D is just a tensor product

$$H^n(\mathcal{C}) \cong H^n(\mathfrak{g}) \otimes C^\infty(\widetilde{M})$$

from which we set $n = 0$ to recover the 0 ghost number cohomology

$$H^0(\mathcal{C}) \cong C^\infty(\widetilde{M})$$

This proves reduction down to the leaf space $\widetilde{M}$ starting from M when taking zeroth cohomology.

2.4.4 The (super) extended phase space and BRST charge: (M_{ext}, Ω)

The BRST complex consists of phase space functions, ghosts and ghost momenta. There are exactly as many ghosts as there are ghost momenta (this is probably only true for coisotropic reduction), which suggests that one should declare ghost and ghost momenta to be conjugate to each other i.e. to extend the bracket structure of the original phase M space to include ghosts $c^i \in \mathfrak{g}^*$ and ghost momenta $b_i \in \mathfrak{g}$. For bosonic phase space, ghosts are anticommuting Grassmann variables. In general, we have graded Poisson brackets between ghosts and antighosts (b_i, c^i) associated to constraints ϕ_i

$$\{b_i, c^j\} = -(-1)^{(\epsilon_i+1)(\epsilon_j+1)}\{c^j, b_i\} = -\delta_i^j.$$

where ϵ_i is the Grassmann parity of the constraint function ϕ_i.

The algebra of superfunctions on the extended phase space M_{ext} that are polynomial in c^i and b_i is the tensor product algebra $\mathbb{C}[b_i] \otimes C^\infty(M) \otimes \mathbb{C}[c^i]$. The BRST charge Ω is the generator of the BRST transformation/derivation i.e. $D = \{\Omega, .\}$. Following (19), it is an odd superfunction that is a sum of an antighost number 0 term and and an anti-ghost number 1 term

$$\Omega = c^i \phi_i - \frac{1}{2} f^i_{jk} c^j c^k b_i. \tag{20}$$

The antighost number 0 term is a linear span of first class functions ϕ_i, while the antighost number 1 term is needed to kill the square of the first term $c^i c^j \{\phi_i, \phi_j\}$ in the Poisson bracket of Ω with itself.

This concludes our discussion of classical BRST starting from constraints. We are now ready to discuss quantization. But before that, let's quickly summarize the important ingredients that went into understanding classical BRST.

Physics	**Geometry**	**Algebra**
phase space	symplectic manifold M	$C^\infty(M)$
constraints	constraint submanifold M_0	vanishing ideal I
gauge symmetries	Hamiltonian vector fields due to first class constraints	ghosts in CE dg algebra
reduction to physical DOF	Poisson reduction to $\widetilde{M}$	zeroth cohomology of BRST complex

2.5 Quantum BRST

Constrained systems and gauge systems in particular abound in nature. However, their reduction destroys properties of description of physical systems we love like manifest covariance, locality, gauge invariance etc. At the same time, physics is ultimately gauge independent and one needs to gauge fix and perform concrete computations. This philosophy of postponing reduction as much as possible is where BRST shines. This is even more useful during quantization as there aren't good tools to perform quantization of reduced systems.

The two most popular ways of doing quantum BRST are

1. **Lagrangian BV** - this is the souped up version of the Faddeev-Popov path integral quantization where constraints don't necessarily close as a Lie algebra (structure constants replaced by structure functions). The BRST symmetry is a symmetry of the action possibly upto equation of motion terms in the most general case. This means that unlike Hamiltonian BRST, off-shell nilpotency is not guaranteed.

2. **Hamiltonian BFV** - this is the canonical/operator based approach to quantize classical gauge systems. The manifestation of the BRST symmetry is a nilpotent BRST operator whose zero ghost-number cohomology picks out the physical Hilbert space.

Since we will focus of Yang-Mills theory in this report, we will not need the full power of BV. We have already explained Faddeev-Popov in the introduction, so we will now turn to BFV.

2.5.1 BFV-BRST quantization

BFV refers to the canonical/operator based quantization of gauge theories that was developed by Batalin, Fradkin, Fradkina and Vilkovisky in the 1970s [FV75, FF78, BV77]. It relies on the-Hamiltonian/phase-space formulation of classical systems. It generalizes Dirac's simple quantization scheme to the case where there are gauge symmetries. There are powerful generalizations of this approach to second-class constraints aka BFF [BFF89] as well as non-trivial curved backgrounds [BFF90, FL94]. Following [FL94], we outline the key steps in this approach

1. *Extension of phase space to super-phase space*: Our original phase space M consisting of position and momenta (q^i, p_i) has non-physical degrees of freedom. To cancel them, we add anticommuting Grassmann pairs of ghost and ghost momenta $(c^\alpha, \mathcal{P}_\alpha)$ corresponding to every independent first class constraint ϕ_α. This is so called the *minimal extension*. However, one can extend it further by including Lagrange multipliers, corresponding non-propagating momenta $(\lambda^\alpha, \pi_\alpha)$ as well as anti-ghosts and their momenta $(\mathcal{P}^\alpha, \bar{c}_\alpha)$. This is known as the *non-minimal extension*. It is a cohomologically trivial extension that may be useful to preserve some manifest symmetries of the system.

2. *Promotion to quantum operators and a natural grading*: We next promote all the phase space variables to quantum operators while Poisson brackets become quantum commutator or anti-commutator depending on the Grassmann parity. Furthermore, we define a Hermitian ghost number operator $\hat{\mathcal{G}}$ whose eigenvalue counts the difference between the number of ghosts and anti-ghosts. With the ghost number, we have a natural grading on the expansion of the quantum observables in powers of ghosts and anti-ghosts. The ghost number zero piece corresponds to the physical sector.

3. *Construction of the BRST operator*: We next look for a nilpotent, ghost number one operator $\hat{\Omega}$ in the space of quantum observables. This is solved order by order in an expansion in powers of ghosts and anti-ghosts. This operator knows about all the gauge symmetries of the system and all gauge-equivalent classes of operators and states are organized by it's cohomology classes.

4. *Operator Cohomology*: To compute the operator cohomology (observables) in the space of quantum operators, we consider the adjoint action $\left[\hat{\Omega}, .\right]$ of the BRST operator $\hat{\Omega}$. Then the cohomology is the quotient
$$H(\hat{\Omega}, \mathcal{F}_{ext}(\hbar)) = \frac{\mathrm{Ker}(\mathrm{ad}\,\hat{\Omega})}{\mathrm{Im}(\mathrm{ad}\,\hat{\Omega})}$$
where $\mathcal{F}_{ext}(\hbar)$ is the algebra of quantum operators built on top of the extended phase space. The physical gauge-invariant observables correspond to the zero ghost number cohomology.

5. *State Cohomology*: Given some representation of $\mathcal{F}_{ext}(\hbar)$ by operators in the Hilbert space $\mathcal{R}_{ext}$, we can compute the corresponding state cohomology
$$H(\hat{\Omega}, \mathcal{R}_{ext}) = \frac{\mathrm{Ker}(\hat{\Omega})}{\mathrm{Im}(\hat{\Omega})}$$
whose zero ghost number states correspond to physical gauge-invariant states.

In the next section, we will illustrate the two quantization procedures through examples: the relativistic particle using BFV and Yang-Mills using Faddeev-Popov.

2.6 Examples

2.6.1 Relativistic particle

A free relativistic particle with rest mass m is constrained by the energy-momentum conservation law $p^2 = -m^2$. We will start from first principles action formulation, identify constraints and construct the extended action. We will thereafter discuss the BRST symmetry and use it to extract physical states. The discussion on constraints and quantum BRST will be based on [Pol98].

The dynamics of a relativistic point particle is obtained by extremizing the following action that computes the geodesic arc length
$$S[x(\tau)] = -m \int \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} d\tau = -m \int \sqrt{-\dot{x}^\mu \dot{x}_\mu} d\tau.$$

Constraint Analysis

We begin the constraint analysis by first calculating the momenta
$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}},$$
which turn out to satisfy the first class constraint $\phi = p^\mu p_\mu + m^2 \equiv 0$ as expected. It then follows that the canonical Hamiltonian is trivial $H = p_\mu \dot{x}^\mu - L = 0$, implying frozen physical dynamics. This is typical of gauge theories. Following Dirac, we then promote H to the extended Hamiltonian H_E which allows dynamics along gauge orbits
$$H_E = \frac{e}{2}\phi = \frac{e}{2}(p^2 + m^2),$$
where e is some Lagrange multiplier. The role of e is to simply parameterize the gauge freedom. To pass to the extended action, we now have to Legendre transform back. By Hamilton's equation, we can express the momenta in terms of "normalized" velocities
$$\dot{x}^\mu = \frac{\partial H_E}{\partial p_\mu} = ep^\mu \implies p_\mu = \frac{\dot{x}_\mu}{e}.$$
The extended action then reads

$$S_E[e(\tau), x(\tau)] = \int \Big(p_\mu \dot{x}^\mu - \frac{e}{2}(p^2 + m^2)\Big) d\tau = \int \frac{1}{2}\Big(\frac{\dot{x}^\mu \dot{x}_\mu}{e} - em^2\Big) d\tau.$$

The above form of the action we just derived is what Polchinski writes down in [Pol98]. We have already commented on the role of e, but there is a nice geometric interpretation as well. Looking at it's equation of motion

$$e^2 = -\frac{\dot{x}^\mu \dot{x}_\mu}{m^2},$$

we see that e behaves as an einbein i.e. it is the square root of some worldline metric. While the worldline metric does not dictate the physical dynamics, it captures the gauge redundancy associated to parameterizing wordline trajectories. The gauge invariance is then precisely the worldline reparameterisation invariance.

Classical BRST

We extend the phase space by adding conjugate ghost and ghost momentum $(c, \mathcal{P})$ corresponding to the first class constraint ϕ. They are conjugate Grassmann odd variables with Poisson bracket

$$\{c, \mathcal{P}\} = -1.$$

Since there is only one first class constraint, the BRST generator Ω is

$$\Omega = c\phi$$

We should note that Ω is indeed nilpotent, the super Poisson bracket $\{\Omega, \Omega\} = 0$ as ϕ commutes with c and $\{c, c\} = 0$. We now wish to find observables that live in the cohomology of Ω. Any observable is a superfunction

$$f = f(q, p, b, c) = f_0(q, p) + \mathcal{P} f_1(q, p) + c f_2(q, p) + \mathcal{P} c f_3(q, p).$$

The action of Ω on f is

$$\Omega f = \{\Omega, f\} = c\{\phi, f_0\} - \phi f_1 + c\mathcal{P}\{\phi, f_1\} - c\phi f_3 \tag{21}$$

So states in the kernel must have $f_1 = 0, \{\phi, f_0\} = \phi f_3 \implies f_0 = \phi \tilde{f}_0, f_3 = \{\phi, \tilde{f}_0\}$. Thus a generic element in the kernel looks like

$$f = \phi \tilde{f}_0 + c f_2 + \mathcal{P} c\{\phi, \tilde{f}_0\} \tag{22}$$

Combining (22) and (21), we compute cohomology (in all possible degrees)

$$H(\Omega) = \frac{(\phi. \oplus \mathcal{P}c\{\phi, .\})C^\infty(M) \oplus cC^\infty(M)}{(\phi. \oplus \mathcal{P}c\{\phi, .\})C^\infty(M) + c(\{\phi, C^\infty(M)\} + \phi C^\infty(M))} \approx c\frac{C^\infty(M)}{(\phi, \{\phi, .\})} \approx c\frac{C^\infty(M_0)}{\{\phi, .\}} \approx cC^\infty(\widetilde{M}) \tag{23}$$

Thus we find that there is non-trivial cohomology only at ghost number 1. It is nice to see the double quotient at action in this example, observables on the original symplectic manifold M are first hit by the constraint ϕ reducing it to the constraint submanifold M_0. This is followed by another quotient by Poisson derivation of the same constraint, reducing the physical system down to the leaf space $\widetilde{M}$.

Quantum BRST

The quantum analysis is almost identical to the classical counterpart (only phase space variables are additionally promoted to operators). So we now wish to focus on physical states in the Hilbert space (state cohomology). There are two steps: labelling states in the Hilbert space via some irreducible representation and then looking for states that belong the cohomology of the quantum BRST charge $\hat{\Omega}$.

Since the $(\hat{p}, \hat{q})$ commute with $\hat{c}$ we can treat the physical and the ghost representations independently. The physical states are labelled by 4-momenta k^μ while the ghost states are labelled by spin $\uparrow$ and $\downarrow$. The complete set of states in total is then labelled by $|\uparrow, k\rangle, |\downarrow, k\rangle$ which are determined in terms of representation of the $\hat{p}^\mu$ and $\hat{b}, \hat{\mathcal{P}}$ operators.

$$\begin{aligned} \hat{p}^\mu |\uparrow, k\rangle = k^\mu |\uparrow, k\rangle, \quad \hat{p}^\mu |\downarrow, k\rangle = k^\mu |\downarrow, k\rangle \\ \hat{\mathcal{P}} |\uparrow, k\rangle = -|\downarrow, k\rangle, \quad \hat{\mathcal{P}} |\downarrow, k\rangle = 0 \\ \hat{c} |\uparrow, k\rangle = 0, \quad \hat{c} |\downarrow, k\rangle = |\uparrow, k\rangle \end{aligned}$$

The action of the BRST charge is

$$\hat{\Omega} |\downarrow, k\rangle = \left(k^2 + m^2\right) |\uparrow, k\rangle, \quad \hat{\Omega} |\uparrow, k\rangle = 0$$

So kernel of $\hat{\Omega}$ is spanned by $|\uparrow, k\rangle$ for any k and $|\downarrow, k\rangle$ for k that satisfy $k^2 + m^2 = 0$. We also find that $|\uparrow, k\rangle = \hat{\Omega}(\frac{1}{k^2+m^2} |\downarrow, k\rangle)$ provided $k^2 + m^2 \neq 0$. Thus the cohomology is

$$H(\hat{\Omega}) = \frac{\text{Ker}}{\text{Im}} = \{\psi \in \text{span}(|\uparrow, k\rangle, |\downarrow, k\rangle) | k^2 + m^2 = 0\}$$

We further note that

$$\langle phy|\uparrow, k\rangle = \delta(k^2 + m^2)$$

where $|phy\rangle$ is any physical state in the cohomology. Since $k^2 + m^2 = 0$ is a hypersurface of measure zero, $|\uparrow, k\rangle$'s existence is kinematically improbable. We conclude

$$H^{phy}(\hat{\Omega}) = \{|\downarrow, k\rangle | k^2 + m^2 = 0\} \tag{24}$$

Comparing (23) and (24), we see that the answers are consistent.

2.6.2 Yang-Mills gauge theory

We have discussed key ideas behind (Faddeev-Popov) BRST quantization of non-abelian gauge theory in the introduction (1). Here we look at constraints and explicitly compute the Faddeev-Popov determinant factor. Note that the Maxwell theory is a special case of Yang-Mills when the gauge group is the abelian group $U(1)$. In Maxwell theory, the ghosts end up decoupling from the gauge bosons and the Faddeev-Popov determinant is just a numerical constant.

Constraint Analysis

The Yang-Mills action is the trace (in adjoint rep) of the square of gauge-covariant curvature 2-form

$$S[A^a_\mu] = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu} F^{\mu\nu}_a$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\nu$ and f^a_{bc} are structure constants of a compact Lie group G. Also we will adopt the mostly positive signature $(-1, 1, 1, 1)$. The action is gauge invariant under the transformation

$$\delta A^a_\mu = D_\mu \varepsilon^a = \partial_\mu \varepsilon^a + g f^a_{bc} \varepsilon^b A^c_\mu.$$

We can count the number of physical degrees of freedom per space time point as the difference between the number of components of the gauge field and twice the number of independent gauge parameters

$$d|G| - 2|G| = |G|(d-2).$$

So in 4 dimensions we have $2|G|$ degrees of freedom. They correspond to the two transverse polarization states per generator of gauge group. For Maxwell theory with $G = U(1)$, one recovers the two transverse photon polarization states while for strong interaction with $G = SU(3)$, one obtains 16 gluon polarization states. Next, we compute the field momenta

$$\pi^0_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_0)} = 0, \quad \pi^i_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^a_i)} = F^{i0}_a = E^i_a$$

We see that there are $|G|$ primary constraints corresponding to the vanishing temporal components

$$\phi^1_a \equiv \pi^0_a \approx 0$$

The canonical Hamiltonian reads

$$\begin{aligned} H_C = \int d^3x \left[\pi^i_a \partial_0 A^a_i + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a\right] &= \int d^3x \left[\pi^i_a (F^a_{0i} + D_i A^a_0) + \frac{1}{2} F^a_{0i} F^{0i}_a + \frac{1}{4} F^a_{ij} F^{ij}_a\right] \\ &= \int d^3x \left[-A^a_0 (D_i \pi^i_a) + \frac{1}{2}\left(\vec{\pi}^2 + \frac{1}{2} F^a_{ij} F^{ij}_a\right)\right] \\ &= \int d^3x \left[-A^a_0 (\vec{D}.\vec{E}_a) + \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)\right] \end{aligned}$$

where $\vec{D}.\vec{E}^a = \vec{\nabla}.\vec{E}^a + g f^a_{bc} \vec{A}^b.\vec{E}^c$, $\quad B^a_i = \frac{1}{2}\epsilon_{ijk} F^{ajk}$. In the second line above, we used integration by parts to shift the gauge-covariant derivative from A^a_0 to π^i_a. We see that the canonical Hamiltonian effectively only depends on the spatial components of the $\vec{E}$ and $\vec{B}$ fields, and the temporal component of the gauge field A^a_0 behaves as a Lagrange multiplier to the transversality condition aka Gauss law $\vec{D}.\vec{E}^a = 0$. This is in fact a secondary constraint given by Poisson bracket of the primary constraints with H_C

$$\phi^2_a \equiv \{H_C, \phi^1_a\} = -(\vec{D}.\vec{E}_a) \approx 0$$

There are no more constraints as ϕ^2_a commutes with H_C. We can thus summarize the constraint algebra

$$\begin{aligned} \{\phi^1_a(\vec{x}), \phi^2_b(\vec{x}')\} = 0, \quad \{\phi^1_a(\vec{x}), \phi^2_b(\vec{x}')\} &= 0 \\ \{\phi^2_a(\vec{x}), \phi^2_b(\vec{x}')\} &= 0 \\ \{H_C, \phi^1_a(\vec{x})\} = \phi^2_a(\vec{x}), \quad \{H_C, \phi^2_a(\vec{x})\} &= 0 \end{aligned}$$

Gauge-fixed path integral

In the introduction, we derived the general form of the gauge-fixing fermion necessary to correctly gauge-fix the action for a gauge field. It is the integral of a ghost number -1 (antighost) expression that is proportional to the gauge condition. To implement the general ξ-gauge, $\partial_\mu A^\mu_a - \frac{\xi}{2} b = 0$, we then choose the gauge fixing fermion Ψ to be

$$\Psi = \int d^4x \; i\bar{c}_a \left(\partial_\mu A^{a\mu} - \frac{\xi}{2} b^a\right)$$

Using (9), we can compute the transformation due to the BRST charge Ω on Ψ as

$$\delta_\Omega \Psi = \int d^4x \; \left(-b_a \partial_\mu A^{a\mu} + \frac{\xi}{2} b^2_a - i\bar{c}_a \partial^\mu D_\mu c^a\right)$$

We thus obtain the gauge-fixed Faddeev-Popov path-integral by adding the BRST-exact piece

$$Z_{gf} = \int [DA^a_\mu Db^a D\bar{c}_a Dc^a] \exp iS_{tot}$$

where

$$S_{tot} = S_0 + \delta_\Omega \Psi = \int \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - b_a \left(\partial^\mu A^a_\mu - \frac{\xi}{2} b^a \right) - i\bar{c}_a \partial_\mu D^\mu c^a \right] \tag{25}$$

One can further integrate out the multiplier field b_a, which amounts to performing a Gaussian integral, resulting in the following standard form of the gauge-fixed path-integral

$$Z_{gf} = \int [DA^a_\mu D\bar{c}_a Dc^a] \exp i \left[\int -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{(\partial_\mu A^a)^2}{2\xi} - i\bar{c}_a \partial_\mu D^\mu c^a \right] \tag{26}$$

In the above path-integral, $\xi = 0$ corresponds to Lorentz gauge while $\xi = 1$ corresponds to Feynman gauge. The Feynman gauge is generally used in the computation of Feynman diagrams in perturbation theory.

From (26), we see that the Faddeev-Popov operator is $\partial_\mu D^\mu$ which is A^μ_a dependent and hence couples the gauge field with ghosts and anti-ghosts. This means that Feynman diagrams for scattering processes involving gauge bosons and fermions must also account for ghost-ghost and gauge boson-ghost interactions. For Maxwell theory however, the gauge-covariant derivative D^μ reduces to the ordinary partial derivative ∂^μ since structure constants are zero. This is the reason why ghosts decouple in the abelian theory and can be integrated out.

Physical states in BRST cohomology

The main point of BRST is to cleanly separate the physical states from unphysical/null states. In the case of Yang-Mills theories, we found that the states in physical Hilbert space must satisfy the first class Gauss law constraint $\vec{D}.\vec{E}_a = 0$. This transversality condition on the admissible polarization states of the gauge boson is also implemented by BRST. Corresponding to every gauge generator, longitudinal and temporal degrees of freedom of the gauge bosons turn out to be BRST exact (null state) and hence die at the level of cohomology. That longitudinal and temporal degrees of freedom are BRST exact can be easily seen from (9) (image of the anti-ghost). The lagrange multiplier b_a which is exact, encodes the unphysical temporal and longitudinal degrees of freedom. To see this, we use (25) to obtain the classical equation of motion of b^a

$$\partial^\mu A^a_\mu = \xi b^a$$

which demonstrates exactness of all states with $k^\mu A^a_\mu \neq 0$. These are exactly the unphysical states we wanted to get rid of, the ones with polarization along the momenta. Thus the BRST cohomology correctly picks only those states that possess transverse helicity.

About a decade into the flurry of work where physicists studied and applied BRST quantization to increasingly general theories, various group of mathematicians got interested in developing quantization more rigorously. The result of their efforts was Kostant-Souriau's geometric quantization [Sni12] on one hand and generalization of the Wigner-Weyl-Moyal phase-space quantization to arbitrary curved symplectic manifolds aka deformation quantization on the other hand. We will next discuss deformation quantization and its geometric realization due to Fedosov.

3 Deformation Quantization

Deformation quantization is an observable first approach to quantization that was first formulated in the 1980s [BFF+78a, BFF+78b, dWL83]. It is the process of *formally deforming* the commutative Poisson algebra of functions on a symplectic (more generally Poisson) manifold into a *non-commutative* one. By deformation, we really mean deforming the commutative product operation and consequently the Poisson bracket. The three italized phrases are the defining properties that we will explain in some detail below.

Let (M, ω) be a symplectic manifold and $Z = C^\infty(M)[[\hbar]]$ be the linear space of formal power series. Deformation quantization assigns a new non-commutative but associative product $*$ (star product) to Z. A typical element in Z looks like

$$a(x, \hbar) = \sum_{k=0}^{\infty} \hbar^k a_k(x)$$

where x are local coordinates on M.

1. *Formal*: Functions in the deformed algebra are formal power series in some deformation parameter $\hbar$. Their convergence properties are typically ignored.
2. *Deformation*: The new product $*$ in the deformed algebra is such that the leading behavior is commutative. So if
$$c(x, \hbar) = a(x, \hbar) * b(x, \hbar) = \sum_{k=0}^{\infty} \hbar^k c_k(x),$$
then the $*$ product at the leading order looks commutative i.e.
$$c_0(x) = a_0(x) b_0(x)$$
3. *Non-commutative*: The star product also has to satisfy the correspondence principle, i.e. the commutator wrt star product should agree with Poisson bracket at leading order in $\hbar$
$$[a, b]_* := a * b - b * a = i\hbar\{a_0, b_0\} + \dots, \quad \forall\, a, b \in Z$$

To perform deformation quantization one has to essentially come up with some non-commutative star product that satisfies the axioms listed above. These structures can be constructed in many ways but in this note we will focus on a geometric approach due to Fedosov [Fed94]. The key ingredient in his construction is an *abelian connection* on a *Weyl algebra bundle* over the symplectic manifold. The same also automatically proves that any symplectic manifold admits a deformation quantization. In the case of more general Poisson manifolds (degenerate cousins of symplectic), Kontsevich famously proved the formality theorem [Kon03] which in turn establishes the existence of their deformation quantization.

The task in the upcoming sections will be to sketch how an abelian connection (curvature is central) on a Weyl algebra bundle induces a star product.

3.1 Formal Weyl Algebra Bundle

Firstly, Weyl algebra is just a quotient of the universal enveloping algebra of the Heisenberg algebra where one sets the central element of the Heisenberg algebra (viz. $[\hat{q}, \hat{p}]$) to be the unit of the universal enveloping algebra. One can think of the Weyl algebra as a quantization of the symmetric algebra of tangent vectors where the tensor product commutator of vectors is given by a symplectic form. Elements in the Weyl alegbra are all possible polynomials built out of $\hat{p}, \hat{q}$ where $[\hat{q}, \hat{p}] = 1$.

Let us turn to the symplectic manifold M at hand. We want to come up with some Weyl algebra that is defined pointwise in M and is induced by the underlying geometry. The (completed) symmetric power of the cotangent bundle is a natural Weyl algebra bundle on M.

$$W_\hbar = \hat{S}(T^*M)[[\hbar]]$$

Let $x \in M$ be a point. A generic element of the fiber over x, W_x, is a formal power series in tangent space coordinates y^α and $\hbar$

$$a(y, \hbar) = \sum_{k, |\alpha| \geq 0} \hbar^k a_{k,\alpha} y^\alpha$$

where y is a coordinate of the tangent space and we have used the multi-index notation

$$y = (y^1, \ldots, y^{2n}) \in T_x M$$
$$\alpha = (\alpha_1, \ldots, \alpha_{2n})$$
$$y^\alpha = (y^1)^{\alpha_1} \ldots (y^1)^{\alpha_{2n}}$$

Thus sections of the bundle are functions of both points on the manifold and on the local tangent space

$$a = a(x, y, h) = \sum_{k,|\alpha|\geq 0} \hbar^k a_{k\alpha}(x) y^\alpha$$

The space of sections $\Gamma(W_\hbar)$ forms a graded (conventionally one sets $\deg \hbar = 2, \deg y = l$ for $y \in S^l(T^*M)$) associative algebra wrt fiberwise multiplication. But that's not all. There is a natural non-commutative product on them as well. This is the Moyal product $\circ$ defined in terms of the underlying symplectic structure ω

$$\begin{aligned} a \circ b &= \exp\left(-\frac{i\hbar}{2}\omega^{ij}(x)\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(y)b(z)\Big|_{z=y} \\ &= \sum_{k=0}^{\infty}\left(\frac{i\hbar}{2}\right)^k \omega^{i_1 j_1}\omega^{i_2 j_2}\ldots\omega^{i_k j_k}\frac{\partial^k a}{\partial y^{i_1}\ldots\partial y^{i_k}}\frac{\partial^k b}{\partial y^{j_1}\ldots\partial y^{j_k}} \end{aligned}$$

Note that the Moyal product is itself a non-commutative product and is indeed the star product for flat symplectic manifolds. What we are looking for is the curved generalization, whose construction will need some more work.

The formal Weyl bundle we have constructed is quite nice. It knows about local star products, but doesn't have any global information. The goal is to then to pull back this geometric non-commutative structure (local star product) from functions on the tangent directions to functions (formal power series) on M. To do this pullback, we now need a notion of parallel section of the Weyl bundle. This motivates the need for a connection which will be called the *Fedosov connection*.

3.2 Fedosov connection

A connection defines a notion of "parallel". Here we are looking to identify parallel sections of the Weyl algebra bundle. This is so that we can map parallel sections of the Weyl bundle to formal power series on M. A section of the Weyl algebra has an extra (prolonged) fiber degree of freedom that has to be fixed so that it can be naturally identified with functions on M that only depend on the base. Regarding the map, we will however go even further and demand an algebra isomorphism σ (symbol map)

$$\sigma : (W_D, \circ) \to (Z, *)$$

where D is the Fedosov connection and $W_D \subset \Gamma(W)$ is the subalgebra of parallel sections of the Weyl algebra wrt D. Once the isomorphism is established, we will be able to define $*$ using $\circ$

$$a * b = \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b))$$

Now let's turn to the structure of the Fedosov connection itself. It is a 1-form D valued in the endomorphism algebra of the space of sections of the Weyl algebra bundle $\mathrm{End}(\Gamma(W))$

$$D = -\delta + \partial + \left[\frac{i}{\hbar}r, .\right] = \partial + \left[\frac{i}{\hbar}(\omega_{ij}y^i dx^j + r), .\right]$$

where δ is like an exterior derivative along the base (maniold) and interior product along the fiber, ∂ is a fixed symplectic connection on the manifold and $r \in W_3 \otimes \Lambda^1$ is a globally defined 1-form satisfying Weyl normalizing condition $r_0 = 0$.

The above connection has two remarkable properties

- *Existence*: The only piece in the definition of D that is non-trivial is the the global 1-form r. Using arguments along the lines of homological perturbation theory, Fedosov is able to show that such an r uniquely exists in the kernel of δ^{-1}. This in turn proves the existence of the connection D.
- *Abelian*: The curvature of D is central meaning it is proportional to the identity endomorphism. This means ($D -$ some central term) is flat. Flat connections are important in quantization as they ensure that quantum evolution of states happens in a path independent way. The other reason we like flat connection is because they behave like a BRST charge and hence allows us to encode gauge symmetries in a homological way.

Fedosov-like deformation quantization can also be done in odd dimensional symplectic manifolds like contact manifolds. In the odd setting of contact manifolds, the abelian Fedosov connection is replaced by a flat connection whose leading abelian piece is the contact 1-form. This has been shown recently in [HW18, CHW21].

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